MATH 4010 29/01/2015

Tutorial 3

Eg 1: Let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$.
Define $T: C[0,1] \to C[0,1]$ by
$$(Tu)(x) = \int_0^1 K(x,y)\,u(y)\,dy, \quad \forall u \in C[0,1].$$
Show that T is a bounded linear operator and $\|T\| = \max_{x\in[0,1]} \int_0^1 |K(x,y)|dy$

Pf: Set $k(x) = \int_0^1 |K(x,y)|dy$. Since K is cts, so is k.
Then $Tu \in C[0,1]$. It is easy to check T is linear.
Since
$$\|Tu\|_c = \max_{x\in[0,1]} |Tu(x)| = \max_{x\in[0,1]} \left|\int_0^1 K(x,y)u(y)dy\right|$$
$$\leq \max_{y\in[0,1]} |u(y)| \max_{x\in[0,1]} \int_0^1 |K(x,y)|dy = \|k\|_c \|u\|_c$$
where $\|\cdot\|_c$ denotes the standard norm on $C[0,1]$.
Therefore T is bounded and $\|T\| \leq \|k\|_c$.
Now, it suffices to show that $\|T\| \geq \|k\|_c$.
Since $k(x) \in C[0,1]$, then $\exists\, x_0 \in [0,1]$ s.t. $k(x_0) = \|k\|_c$.
Set $\varphi(y) = \operatorname{sgn} K(x_0,y) = \begin{cases} 1 & \text{if } K(x_0,y) > 0 \\ -1 & \text{if } K(x_0,y) < 0 \end{cases}$

By Lusin's Thm, $\exists \{u_n\} \subset C[0,1]$, (w.l.o.g with $|u_n(y)| \leq 1$) s.t. $u_n \to \varphi$
$$(\|k\|_c =) k(x_0) = \int_0^1 K(x_0,y)\varphi(y)dy.$$
$$= \lim_{n\to\infty} \int_0^1 K(x_0,y)u_n(y)dy \quad \text{by L.D.C.T.}$$
$$= \lim_{n\to\infty} (Tu_n)(x_0) \leq \overline{\lim_{n\to\infty}} \|Tu_n\|$$
$$\leq \overline{\lim_{n\to\infty}} \|T\|\|u_n\| \leq \|T\|$$

Hence, T is a bounded linear operator with $\|T\| = \max_{x\in[0,1]} \int_0^1 |K(x,y)|dy$.

Eg 2. Let $A=(a_{ij})$ $(i,j=1,2,\cdots)$ be a infinite matrix.

Define $A: x \mapsto Ax$ by

$$y = Ax \quad x=\{x_j\},\ y=\{y_i\}$$

$$y_i = \sum_j a_{ij} x_j.$$

Then: (i) if $c = \sup_j \sum_i |a_{ij}| < +\infty$, then A is a bounded linear operator on ℓ^1 and $\|A\|_1 = c$

(ii) if $c = \sup_i \sum_j |a_{ij}| < +\infty$, then A is a bounded linear operator on ℓ^∞ and $\|A\|_\infty = c$

(iii) if $c = \left(\sum\left(\sum |a_{ij}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < +\infty$, for $1<p,q<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$

then A is a bounded linear operator on ℓ^p and $\|A\|_p \leq c$.

Pf: (i) Since $\sum_j |a_{ij} x_j| \leq \sup_j |a_{ij}| \sum_j |x_j|$, $\sup_j |a_{ij}| \leq c < +\infty$, $\{x_j\} \in \ell^1 \Rightarrow \sum_j |x_j| < +\infty$, then $\sum_j |a_{ij} x_j| < +\infty$, i.e. y_i is well-defined.

$$\|y\|_{\ell^1} = \sum_i |y_i| = \sum_i \left|\sum_j a_{ij} x_j\right| \leq \sum_i \sup_j |a_{ij}| \sum_j |x_j| \leq c\|x\|_{\ell^1}$$

Hence, A is bounded and $\|A\|_1 \leq c$.

It sufficies to show $\|A\|_1 \geq c$, which is equivalent to prove that $\forall c' < c$, $\|A\|_1 > c'$.

By the definition of c, $\forall c'<c$, $\exists j_0$ s.t. $c' < \sum_i |a_{ij_0}|$

Let $\{e_j = (0,\cdots 0, \underset{\substack{\uparrow \\ j\text{-th}}}{1}, 0, \cdots)^t\}$ be a basis of ℓ^1,

then $Ae_j = (a_{1j}, a_{2j}, \cdots)$

$$c' < \sum_i |a_{ij_0}| = \|Ae_{j_0}\|_{\ell^1} \leq \|A\|_1 \|e_{j_0}\|_{\ell^1} = \|A\|_1.$$

(ii). $\sum_j |a_{ij}||x_j| \leq c\|x\|_{\ell^\infty} < +\infty$, $\forall x \in \ell^\infty$.

and $\|y\|_{\ell^\infty} \leq \sup_i |y_i| \leq \sup_i \sum_j |a_{ij} x_j| \leq \sup_i \sum_j |a_{ij}| \sup_j |x_j| \leq c\|x\|_{\ell^\infty}$

Hence A is bounded and $\|A\|_\infty \leq c$.

It sufficies to show $\|A\|_\infty \geq c$, i.e. $\forall c'<c$, $\|A\|_\infty > c'$.

$\forall c'<c$, $\exists i_0$ s.t. $c' < \sum_j |a_{i_0 j}|$. Set $x_j = \operatorname{sgn} a_{i_0 j}$.

then $\|x\|_{\ell^\infty} = 1 < +\infty$ and

$$c' < \sum_j |a_{i_0 j}| = \sum_j a_{i_0 j} x_j = y_{i_0} \leq \|A\|_\infty \|x\|_{\ell^\infty} \leq \|A\|_\infty.$$

(iii) By Hölder's ineq. $|y_i| \leq \left(\sum_j |a_{ij}|^q\right)^{\frac{1}{q}} \left(\sum_j |x_j|^p\right)^{\frac{1}{p}} < +\infty$, A is well-defined

$$\|y\|_{\ell^p}^p = \sum_i |y_i|^p = \sum_i \left|\sum_j a_{ij}x_j\right|^p \leq \sum_i \left[\left(\sum_j |a_{ij}|^q\right)^{\frac{1}{q}} \|x\|_{\ell^p}\right]^p$$

Then $\|y\|_{\ell^p} \leq \left(\sum_i \left(\sum_j |a_{ij}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \|x\|_{\ell^p} \leq C\|x\|_{\ell^p}$

Hence, A is bounded and $\|A\| \leq C$.